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Parts, Wholes, and Quantity in Euclid's Elements

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Parts, Wholes, and Quantity in Euclid's Elements

This paper develops a novel methodology, combining history of mathematics, philology, philosophy of mathematics, and logic. We develop a formal logical treatment of Euclid's *Elements*, in which set theory plays no role, but the logic of part and whole does. We first consider a controversy about the nature of Euclid's *Elements* Book II. For Euclid, the part-whole relation plays roles that are now played by arithmetic operations. This shows one crucial limitation of the controversial interpretation of this text as geometrical algebra. Returning to the beginning, we present a formal language for stating Propositions 1 through 10 (omitting 7) and proofs of them. Surprisingly, this has never been done (except for one recent approach, which differs from ours in an essential way). We conclude by sketching several significant ways in which this project can be further developed.

Geometry; extension; body; logic; mathematics; transformation.

1 Introduction

In this paper, we investigate the logic of continuous geometric objects and in particular the use of the concepts of *part* and *whole* in reasoning about those objects. We are pursuing two interconnected goals. One is the interpretation of Euclid, and especially the interpretation of his quantitative concepts and his use of the part-whole relation. The other is the development of a logic of continuous objects *that does not presuppose set theory*. To begin with, we sketch how these goals are connected.

The foundations of mathematics are usually conceived in set theoretic terms. Natural numbers, integers, rational numbers, and real numbers are defined in set theoretic terms. Geometric objects are then introduced in terms of sets of n-tuples of real numbers (e.g., the unit-circle centered at the origin is the set of all points (x, y) in $\mathbb{R} \times \mathbb{R}$ that satisfy $x^2 + y^2 = 1$, where $\mathbb{R}$ is the set of real numbers). Predicates such as *three-dimensional* or *continuous* are also defined in set theoretic terms. Surprisingly, there has not been much investigation of the prospects for developing a formal logic of continuous objects that does not presuppose set theory. Why would one want to do this? First, set theory is controversial. Second, (normal versions of) set theory would seem to be a much more powerful theory than one needs. Third, and most important, it is highly plausible that the theory of continuous objects should not presuppose set theory and especially that the theory of continuous objects should not treat points as primitive and objects as collections of points. There is a long tradition, going back to Aristotle, of conceiving of points as derivative from composite objects. At the very least, such theories should be developed and compared with more familiar set-theory-based theories. Where in the history of philosophy and mathematics can we find such theories? Greek geometry is one of the most promising sources of a theory of continuous objects that does not presuppose set theory. Thus one promising approach to developing such a logic of continuous objects is to flesh out (say) Euclid's geometric theory of continuous objects in the *Elements* into a full-fledged formal logic.

The undertaking just described need not have anything much to do with understanding Euclid. But, properly carried out, it can. There are numerous logical gaps (by our lights) in the *Elements*. Rather than filling those gaps in whatever way seems to *us* best, one can try to fill those gaps as Euclid would have. This is, of course, no trivial undertaking. One major question that has arisen in recent discussion of the *Elements* is whether, for Euclid, the diagrams play a role in justifying inferences. The question is not whether the text Euclid wrote presupposes diagrams. Reviel Netz has argued decisively that it does.[1] But Netz's arguments are entirely compatible with the diagrams playing a merely 'heuristic' role, as an aid to the reader's comprehension, rather than an inferential role, justifying inferences from one sentence to another. Our reconstruction of Euclid's demonstrations will work on the assumption that the diagrams play no inferential role. The question that we will try to illuminate is this: to what extent does Euclid rely on the concepts of *part* and *whole*? In order to answer this question, we employ contemporary formal logic as an interpretive tool.

In section 2, we develop further an interpretive problem concerning Euclid, namely how to understand propositions 1 through 10 of Book II (the so-called 'geometric algebra'). This section provides a strong justification for expecting the concepts of *part* and *whole* to play a major role in Euclid's reasoning – in particular, to play a role in connection with quantitative reasoning. In section 3, we develop a logical language and give proofs for the first ten propositions of Book I (except for 7). Section 4 considers how this language might be further developed or improved. Section 5 then explains the significance of our *mereogeometry* and sketches directions for further research.

2 Part and whole in *Elements* book II

Of recent controversy about the *Elements*, some of the most heated has focused on Book II. This book is puzzling, because its first 10 propositions are so straightforward that there might seem to be no point in proving them. One prominent interpretation has explained why Euclid would prove these theorems by characterizing them as *geometric algebra*. On this interpretation, these propositions prove algebraic facts (such as the distributive law), some fundamental, some more complicated and useful for solving quadratic equations. Sabetai Unguru launched a polemical attack on this line of interpretation, arguing that Book II is not algebraic at all, but rather geometric.[2] Ken Saito has elaborated this line of thought via an examination of how the propositions of Euclid II are used in Apollonius.[3] He conceives Euclid II as a 'toolbox' for reasoning about invisible figures. It seems to us that the parties to this controversy have overlooked the question of how *precisely* to understand what Propositions 1 through 10 of Book II say and of how to compare those propositions with their putative algebraic correlates. We think that careful attention to the logic of these propositions provides essential illumination of their meaning, which shows how they are not algebraic in character and also provides a necessary basis for Saito's analysis of their use.

The label 'geometric algebra' is highly problematic (as Ian Mueller noted in his critical discussion of the whole line of interpretation).[4] It has never, to my knowledge, been satisfactorily elucidated, even by those who are fiercely critical of it. One aspect of the thought, however, is fairly clear. It is that the propositions of Euclid II express the same thing as is expressed by familiar algebraic equations. But this is obviously unsatisfactory

1 See Netz 1999, Chapter 1.
2 Unguru 1975.
3 Saito 1985.
4 Mueller 2006, 41–52.

as an account of the meaning of any of the first ten propositions of Euclid II. Consider Proposition 4, for instance (see Fig. 1).

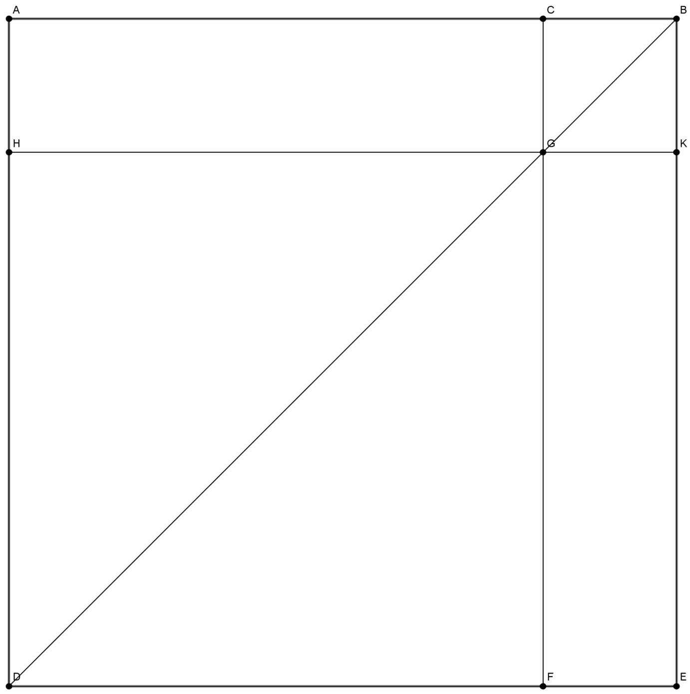

Fig. 1 | Diagramm

It says, "If a straight line be cut arbitrarily, then the square on the whole is equal to the squares on the segments and twice the rectangle contained by the segments."[5] If one interprets the proposition as geometric algebra, then it says the same as $(x + y)^2 = x^2 + 2xy + y^2$. (We will call this algebraic statement 'the binomial formula'). But Euclid's proposition is a proposition about squares and rectangles, i.e., about geometric figures. While it corresponds to the binomial formula, the binomial formula does not say what the proposition says. To establish this, it is enough to note that the proposition is about squares and rectangles, not about real numbers. Of course, with that remark, we assume that the binomial formula is an expression implicitly quantified over the real numbers (For every x, y, such that x and y are real numbers, $(x+y)^2 = x^2+2xy+y^2$). Not only is Proposition II.4 not about (positive) real numbers, Euclid did not have the real numbers at all, not even the positive ones. He has no term that corresponds to the term real number. It has been argued that the Euclidean (or Eudoxan) concept of ratio, as defined and used in Book V, corresponds to the concept of a (positive) real number. Even if this were so (and we deny it), the proposition in question occurs in Book II, long before the concept of ratio is defined or used.

The authors who have oppose the interpretation of Euclid in terms of geometric algebraic also formulate the statement of II.4 as an equation (a statement involving the sign '='). They might write something like[6]:

If a line AB is divided at C, then sq AB = sq AC + sq CB +2 rect AC,CB.

5 The translation is my own.
6 Mueller 2006 does this, for instance, throughout the book.

What does this mean? Well, 'sq AB' refers to the square that one *could* make on the line segment AB. And 'rect AC,CB' refers to the rectangle that one could construct from the line segments AC and CB as sides. There are complications about the meanings of these expressions, but we will not explore those complications here. Rather, I would like to focus on the signs '=' and '+'.

Let us begin with '='. In contemporary philosophy and mathematics, '=' is normally used as a sign for the identity relation. For instance, one might write, 'For all x, $x = x$' to say that every object is identical with itself or 'For all x, $x + 1 = 1 + x$' to say that the result of adding 1 to x is *identical* with the result of adding x to 1. Normally, statements such as '$2 + 2 = 4$' are taken to express identity – that is, the object named '4' is identical with the object that is the value of the addition-function for the arguments 2 and 2.

The identity relation is not, however, the relation of which Euclid speaks in II.4. Euclid's Greek is equipped to mark clearly the distinction between the *identity* relation and the *same-size-as* relation. It is perhaps worth pausing for a moment to impress on ourselves that these are in fact two distinct relations. The *same-size-as* relation is distinct from the identity relation because there are things that are not identical, but are the same size – your two hands, for instance. Of course, everything is the same size as itself (at any given time), but many things are the same size as *other* things, i.e., as things with which they are not identical. Now Euclid expresses the same-size-as relation in Greek by using the expression 'is equal to', whereas he expresses the identity relation simply by using the verb 'to be'. The difference corresponds exactly to the difference between, say, 'The square on AB is *equal* to the square on BC' (speaking of two distinct, equal squares), and 'The square on AB *is* the square on BC' (speaking of a single square that has AB as one side and BC as another, see Figs. 2 and 3).

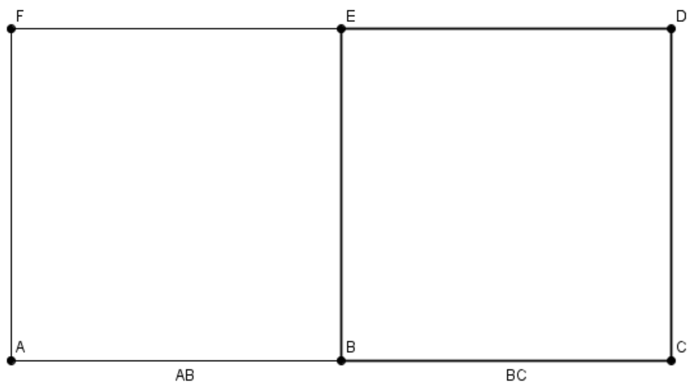

Fig. 2 | Diagramm

Euclid's Proposition II.4 clearly asserts that sq AB is *equal to* (not identical with) something. Thus it behooves not to use the standard sign for identity, '=', but rather another sign. We will use '≡' to mean 'is the same size as'.

Just as important, while many commentators use the sign '+' between expressions of the form 'sq X' or 'rect X,Y', none of them says anything about what it means. This is a non-trivial matter. We would normally treat '+' as referring to the addition function that takes a pair of numbers (natural, rational, real, imaginary) to their (additive) sum. When the '+' sign is used in an expression such as '$(x + y)^2 = x^2 + 2xy + y^2$', then this is the obvious way to understand it. It signifies the addition function. But what should the '+' sign signify in the expression:

sq AB ≡ sq AC + sq C +2 rect AC,CB

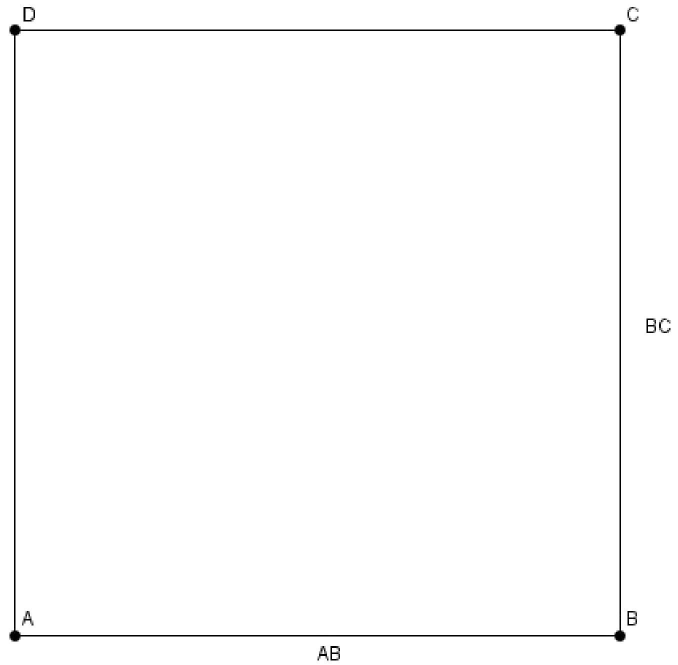

Fig. 3 | Diagramm

One possibility is that the terms for geometric figures ('sq AC' etc.) stand for areas, i.e., for real numbers, to which an area function maps the given figures. Then '+' would have a familiar and well-defined meaning, namely the addition function on the reals. But this is obviously not what is intended by the commentators who use '+'. For the whole point of writing the above expression was to avoid the use of the real numbers and to remain, rather, with geometric figures.

Given that we interpret Euclid II.4 in terms of geometric figures, then there is a serious question about the meaning of '+'. What does it mean to add two geometric figures, if it does *not* mean adding their areas? One thing it might mean is that the figures are to be put together, boundary on boundary, without overlapping or leaving any gap, so as to form another figure. This third figure, of which both initial figures are parts, would be their sum. (See Figure 4.) But this is quite clearly not what Euclid means in the propositions in question. And Euclid often uses the *same-size-as* relation without concatenating the figures in question. Consider, for instance, Euclid's formulation of the Pythagorean theorem (Book I, Proposition 47): "In right-angled triangles, the square on the side subtending the right angle is equal to the squares on the sides containing the right angle." Some translators, such as Fitzpatrick, insert a phrase such as 'the sum of', so that the theorem reads, 'In right-angled triangles, the square on the side subtending the right angle is equal to *the sum of* the squares on the sides containing the right angle' (emphasis added). Whatever 'the sum of two squares' means, it cannot mean 'a figure composed out of two squares'. For Euclid constructs the squares on the sides containing the right angle in such a way that they do not compose a single figure. (Their boundaries meet at a single point and thus the definition of a figure does not apply.) There is nothing in the diagram or in the demonstration that corresponds to the composition of the two squares. What, then, does '+' or 'sum' mean here?

Rather than answering this question, we propose a new way of understanding the *same-size-as* relation. Unlike the identity relation, this relation might not be a straightforward two-place relation. Of course, sometimes it is used to assert the sameness of size of two items, but it can also be used to assert that one item is the same size as two other items – not severally but, as translators sometimes write, 'taken together'. 'Taken together' does not express the application of a function or operation, but just makes clear that the

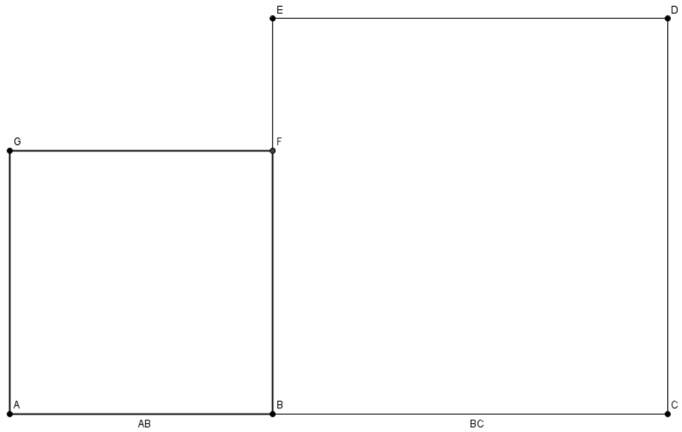

Fig. 4 | Diagramm

items belong *together* on one side of the *same-size-as* relation. Thus Euclid's formulation of the Pythagorean theorem makes no use of addition. It can be better written:

> Square on side subtending right angle ≡ square on one side containing right angle, square on other side containing right angle

In other words, the sign '+' is quite misleading. No operation whatsoever is performed on the squares in question. The equality relation simply applies directly to them. In the same way, Euclid II.4 should be written without '=' and without '+', thus:

> sq AB ≡ sq AC, sq CB, 2 rect AC,CB

Note that the right-hand side can hardly be a mereological sum, since it has *twice* the rectangle AC,CB. (The mereological sum of Socrates and Socrates is just Socrates.) This is not, however, to say that Euclid makes no use of concatenation (roughly, putting together figures in such a way that they compose another figure, of which they are parts). It is just to say that concatenation is not used in the statements of theorems such as the Pythagorean theorem.

But Euclid does often use dissection and concatenation of figures. It is in this context that he uses the concepts of *part* and *whole*, which are the focus of our interest. Perhaps the first mathematically interesting theorem in which this occurs is Book I, Proposition 35: "Parallelograms which are on the same base and between the same parallels are equal to one another."

The proof proceeds as follows (see Fig. 5). First, triangles ABE and DCF are proved to be congruent to one another. Now, a dissection is performed. The triangle DGE, which is precisely the overlap between triangles ABE and DCF, is removed and the remainders (two trapezoids) are therefore equal. What happens here is quite different from an ordinary algebraic operation (the rewriting of an equation containing variables that stand for real numbers). We have two continuous geometric figures. We observe that each figure is composed of two parts, a trapezoid and the triangle DGE, and on this basis infer that the trapezoids are equal. In algebra, there are no continuous wholes of this kind. Euclid is relying on the common notion that says if equals are taken from equals, then the remainders are equal. But this common notion is a statement about wholes and parts – in particular, that if two parts of two equal wholes are equal, then the remainders are also equal. It is thus very different from the rules for manipulation of equations in algebra. These rules are based on facts about binary functions on the real numbers. The concepts of part and whole play no role.

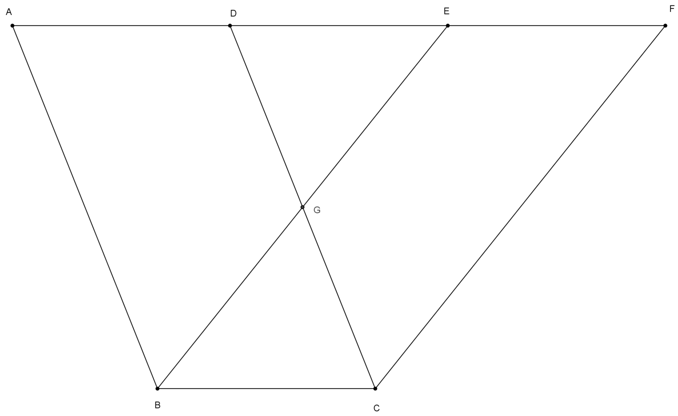

Fig. 5 | Diagramm

A major goal of our project is to clarify the assumptions about parts and wholes that underlie Euclid I.35 and, indeed, all of Euclidean geometry. This paper is but a small first step in that project. Roughly, Euclid seems to rely on three assumption. First, he assumes lines bounding the parallelograms do not have any area.

Second, he assumes that if a figure has an internal boundary (the line DG in the triangle ABE, the line GE in the triangle DCF), then it is composed of two figures, each of which has that boundary as an external boundary. This assumption is not terribly bold or surprising. It just says that if a figure is divided by a line or a plane, then that line or plane divides it *into two figures*.

Third, Euclid relies on the opposite assumption. He composes the triangle BGC with each of the trapezoids. Since the trapezoids were equal, and each is being composed with one and the same triangle BGC, the resulting figures, i.e., the parallelograms $ABCD$ and $BEFC$, are equal. Again, this is not an operation on numbers, but on figures. And the operation is not the application of a binary function to a set of objects, all of which could equally be arguments of the function, but is rather the application of a principle of composition. If two non-overlapping figures share a line as a boundary, then there is a third figure composed of both of them. This assumption is also unsurprising and entirely reasonable. But it is not an assumption that interpreters have identified and discussed.

So there are very strong reasons to think that Euclid relies on the concepts of *part* and *whole* in his reasoning about figures and their sizes. But in order to give a precise and rigorous account of how this works, we need to return to the beginning of the *Elements* and determine precisely where and how its demonstrations rely on these concepts. Otherwise, it might well be that Euclid does not in fact, in any essential way, rely on those concepts in Book II, even if he seems to.

3 Formal proofs of *Elements* I.1–3

In this section, we begin the task of giving proofs in formal languages. (We are using 'demonstration' for a section the Euclidean text (the *apodeixis*), reserving 'proof' for a sequence of sentences that is a proof *by the standards of contemporary logic*. That is, 'demonstration' refers to what is called in Greek the *apodeixis*, in the Proclan analysis of a proposition into enunciation, setting forth, determination, construction, demonstration, and conclusion. Thus the demonstrations (Euclid's writing) are not necessarily proofs.) We will use the language of first-order predicate logic and supplement it with a number of predicates. Even the determination of these predicates is a non-trivial matter.

There are two aspects of Euclid's reasoning that we will neglect. First, he always proceeds by ekthesis. That is, having stated the result to be proved in general terms, he introduces a particular instance ('Let there be a triangle ABC'), carries out the construction and demonstration, and then infers the general result. While this is an important and interesting aspect of Euclid's procedure, we neglect it because it would introduce major complications and it does not seems necessary in order to answer our immediate questions about parts and wholes. Second, Euclid's formulations of propositions fall into two groups, so-called problems and theorems. The theorems are expressed in truth-valuable assertoric sentences. The problems, by contrast, are expressed with infinitival clauses. The first proposition of the *Elements* is: "On a given a given bounded straight line, to construct an equilateral triangle". We are much interested in the distinction between theorems and problems (and it is the topic of a work in progress by Jonathan Beere and Ben Morison), but for our purposes in this paper, we will treat problems as ∀∃-sentences. So, for instance, we will treat the proposition just mentioned as, "For every bounded straight line, there exists an equilateral triangle with that straight line as side".

3.1 Language and axioms for stating *Elements* I.1

Let us begin at the beginning, with the demonstration of the proposition just noted. This proposition uses the predicates, '...is a straight line', '...is an equilateral triangle', and '...is on ...'. We should begin by considering the question of how to define these notions, if at all. How does Euclid define them?

He defines a straight line as a line "that lies evenly [ἐξ ἴσου] with the points on itself" (I, def. 4). This definition is perhaps the most obscure that Euclid gives, and he gives no explicit axioms that involve the predicate 'lies evenly'. More helpful is the previous definition, which says about all lines (not only straight ones) that their extremes are points. We will assume that, for each straight line, there are precisely two of them, which we will call 'endpoints of s'. The first postulate of the *Elements* says (in our revised version), "For every pair of points, A and B, there is a straight line from A to B''. We take this to mean that A and B are the endpoints of the straight line.

Euclid will also assume that there is *only one* straight line between a given pair of points. Euclid already makes this assumption in his proof of I.4 (which corresponds to the theorem now commonly known as SAS). Thus we might formalize our strengthened version of Postulate 1 in two ways. First:

$\forall xy$ (x is a point and y is a point and $x \neq y \rightarrow \exists_1 z$ (z is a straight line from x to y))

Or alternatively:

$\forall xy$ (x is a point and y is a point and $x \neq y \rightarrow \exists_1 z$ (z is a straight line and x is an endpoint of z and y is an endpoint of z))

The difference is that the first version uses a three-place predicate ('...is a straight line from ...to ...') whereas the second version uses a one-place predicate and a two-place predicate ('...is a straight line' and '...is an endpoint of ...'). Which version should we prefer?

To begin with, note that the three-place predicate can be defined in terms of the other two, or the other two in terms of the three-place predicate. Assuming the three-place predicate, we can introduce the following definitions:

z is a straight line $:\leftrightarrow \exists xy$ ($x \neq y$ and z is a straight line from x to y)

x is an endpoint of z $:\leftrightarrow \exists y$ ($x \neq y$ and (z is a straight line from x to y or z is a straight line from y to x))

Assuming the one-place and two-place predicates, we can define the three-place predicate as follows:

z is a straight line from x to $y :\leftrightarrow x \neq y$ and z is a straight line and x is an endpoint of z and y is an endpoint of z

Which definition constitutes the superior interpretation of the Euclidean text? Definition 4 from the *Elements* seems to define 'straight line' as a one-place predicate ("a straight line is one that lies evenly with the points on itself"). On the other hand, Definition 3 says that the extremities of a line (any line, whether curved or straight) are points and this suggests that the predicate may have more than one place. Moreover, Euclid normally (but not always) refers to lines by two end-points, as at the beginning of the demonstration of I.1, where he says, "Let AB be a given straight line." This speaks in favor of the three-place predicate.

Furthermore, we incline to think that using a three-place predicate is the more elegant way to develop our formal language. It allows us to formulate many propositions more compactly, using a single three-place predicate, rather than three conjoined sentences (corresponding to the definition above).

Similar considerations apply to the predicate 'triangle.' Euclid defines a triangle as, "a rectilinear figure contained by three straight lines" (I, def. 19) and an equilateral triangle as a triangle that has three equal sides (I, def. 20). As a straight line is determined by 2 objects that are not themselves straight lines – its endpoints – so a triangle is determined by 3 objects that are not themselves triangles – its straight-line sides. By analogy with our three-place predicate '…is a straight line from …to ..', we will work with a four-place predicate, 'Triangle …is determined by …, …, ..' But here we have a choice between defining triangles as determined by their vertices or by their sides. That is: are the three determining objects points or lines?

The definition of triangle just cited suggests that triangles are determined by their sides, not by their vertices. But Euclid's way of proceeding in his demonstrations suggests the contrary. For he refers to triangles by way of their vertices, in expressions such as, 'The triangle ABC'. We will give priority to Euclid's procedure in the demonstrations rather than to his explicit definition, and define triangles in terms of their vertices. Hence we use the predicate:

x is a triangle with vertices a, b, and c

Thus the definition of *equilateral* triangle takes the following form: the triangle with vertices a, b, and c is equilateral if and only if the straight lines between the three pairs of points are equally long. Thus we also need a predicate '…is as long as ..' In Section 2, we already argued that Euclid will use a similar equality-predicate for figures. Here, we are introducing such a predicate for lines.

Thus far, our language L[E] contains the following predicates:

Px	x is a point
$x \equiv y$	Straight line x is equal to (as long as) straight line y
Ssxy	s is a straight line from x to y
Δ*xabc*	x is a triangle with vertices a, b, c

Thus Proposition 1 is written as follows in L[E] as developed so far[7]:

7 For the reader who is innocent of logic, the following guide might be useful. '$\forall abc$' is read 'for every a,b,c.' '$\exists abc$' is read 'there exist a,b,c such that ..' '$a = b$' is read 'a is identical with b.' '$\rightarrow$' is always flanked by statements; it is read 'if [first statement] then [second statement].' Following the usual convention, we write the predicates before the terms they apply to (e.g., 'Sx' is read 'x is a straight line'). '$\vdash$' may be read 'it is a theorem that ..' If flanked by lists of statements, it is read '[statements on left] entail [statement on right].'

$$\forall xab\,(Sxab \to \exists y\,(\Delta yabx \wedge y \text{ is an equilateral triangle} \wedge y \text{ is on } x))$$

This formalization fits well with Euclid's actual way of proceeding in Proposition 1. Informally, one would begin a proof of this proposition by saying, 'Let a and b be two arbitrary points and s the straight line between them.' Setting aside mere notational differences, this is precisely how Euclid begins.

The predicate 'y is on x' is not defined by Euclid. We assume that this predicate means the same as 'x is a side of y' (if y is a triangle and x is a straight line). We can then rewrite Proposition 1:

$$\forall xab(Sxab \to \exists a'b'c'yss's''(\Delta ya'b'c' \wedge Ssa'b' \wedge Ss'a'c' \wedge Ss''b'c' \wedge s \equiv s' \wedge s \equiv s'' \\ \wedge\, s' \equiv s'' \wedge (x = s \vee x = s' \vee x = s'')))$$

By symmetry, it is irrelevant which side of the triangle y is the given straight line x. We can thus stipulate without loss of generality that x is s, which simplifies the formula as follows:

$$\forall xab(Sxab \to \exists a'b'c'yss's''(\Delta ya'b'c' \wedge Ssa'b' \wedge Ss'a'c' \wedge Ss''b'c' \wedge s \equiv s' \wedge s \equiv s'' \\ \wedge\, s' \equiv s'' \wedge x = s))$$

But this is obviously equivalent to:

$$\forall xab(Sxab \to \exists a'b'c'ycs's''(\Delta ya'b'c' \wedge Sxa'b' \wedge Ss'a'c' \wedge Ss''b'c' \wedge x \equiv s' \wedge x \equiv s'' \\ \wedge\, s' \equiv s''))$$

This formulation could also be further simplified, should we obtain suitable axioms for the expressions used in it, as we will go on to do.

We begin by stating a number of natural axioms for the predicates 'P', 'S', and '$\equiv$'. We will use 'S' as a one-place predicate with the definition (stated above)

$$Sz :\leftrightarrow \exists xy\, Szxy.$$

(Ax 1) $\forall xys\,(Ssxy \to Px \wedge Py)$

(Ax 2) $\forall xys\,(Ssxy \to x \neq y)$

(Ax 3) $\forall xys\,(Ssxy \to Ssyx)$

(Ax 4) $\forall xyss'\,(Ssxy \wedge Ss'xy \to s = s')$

(Ax 5) $\forall xy\,(Px \wedge Py \wedge x \neq y \to \exists sSsxy)$

(Ax 6) $\forall xyabs\,(Ssxy \wedge Ssab \to (x = a \wedge y = b) \vee (x = b \wedge y = a))$

(Ax 7) $\forall xy\,(x \equiv y \to Sx \wedge Sy)$

(Ax 8) $\forall x\,(Sx \to x \equiv x)$

(Ax 9) $\forall xy\,(x \equiv y \to y \equiv x)$

(Ax 10) $\forall xy\,(x \equiv y \wedge y \equiv z \to x \equiv z)$

To investigate the role of the part-whole relation, we will need an '…is a part of …' predicate, '$\sqsubseteq$'. For the moment, we introduce the following purely mereological axioms:

(Ax 11) $\forall x\, (x \sqsubseteq x)$[8]

(Ax 12) $\forall xyz\, (x \sqsubseteq y \wedge y \sqsubseteq z \rightarrow x \sqsubseteq z)$

(Ax 13) $\forall xy\, (x \sqsubseteq y \wedge y \sqsubseteq x \rightarrow x = y)$

In addition, we will introduce the following axioms that employ both mereological and geometric predicates:

(Ax 14) $\forall x\, (Px \rightarrow \exists y\, (Py \wedge x \neq y))$

(Ax 15) $\forall xys\, (Ssxy \rightarrow x \sqsubseteq s \wedge y \sqsubseteq s)$

(Ax 16) $\forall xyst\, (Ss \wedge Stxy \wedge x \sqsubseteq s \wedge y \sqsubseteq s \rightarrow t \sqsubseteq s)$

Finally, we need axioms governing the predicate Δ. These axioms say that there is a triangle between three points if and only if the points are not collinear. Thus we need a three-place predicate for collinearity:

Definition: $Coll(abc) :\leftrightarrow Pa \wedge Pb \wedge Pc \wedge \exists s\, (Ss \wedge a \sqsubseteq s \wedge b \sqsubseteq s \wedge c \sqsubseteq s)$

(Ax 17) $\forall abcx\, (\Delta xabc \rightarrow Pa \wedge Pb \wedge Pc \wedge \neg Coll(abc))$

(Ax 18) $\forall abc\, (Pa \wedge Pb \wedge Pc \wedge \neg Coll(abc) \rightarrow \exists x\, \Delta xabc)$

Using these axioms and definitions, we prove the following four-part lemma:

Lemma 1: (i) (Ax 6) $\vdash \forall xyzst\, (Ssxy \wedge Stxz \wedge s = t \rightarrow y = z)$
(ii) (Ax 5, 14, 15) $\vdash \forall x\, (Px \rightarrow \exists y\, (Sy \wedge x \sqsubseteq y))$
(iii) (Ax 5, 14, 15) $\vdash \forall xyz\, (Px \wedge Py \wedge Pz \wedge \neg Coll(xyz) \rightarrow x \neq y \wedge x \neq z \wedge y \neq z)$
(iv) (Ax 5, 14, 15) $\vdash \forall xyz\, (Px \wedge Py \wedge Pz \wedge \neg Coll(xyz) \rightarrow \exists s\, Ssxy)$

Proof: (i) By (Ax 6).
(ii) By (Ax 5), (Ax 14) and (Ax 15).
(iii) By (ii) and (Ax 5).
(iv) By (iii) and (Ax 5).

Relative to these axioms, the formalization of Proposition 1 from the end of the last section is equivalent to:

$$\forall xab\, (Sxab \rightarrow \exists ycss'\, (\Delta yabc \wedge Ssac \wedge Ss'bc \wedge x \equiv s \wedge x \equiv s'))$$

This is the version that we will now go on to prove.

3.2 Circles

Early in the demonstration of I.1, Euclid constructs two circles, using postulate 3, one having A as center with B on its circumference, the other having B as center with A on its circumference. See Figure 6. Euclid assumes that the two circles intersect.

8 Euclid would surely not have accepted this axiom. In this first version, we use it because it has been a part of standard mereology. In a future revised version, we hope to avoid this axiom. On the other hand, it does not seem to cause any serious problem, because one can easily restrict parts to proper parts.

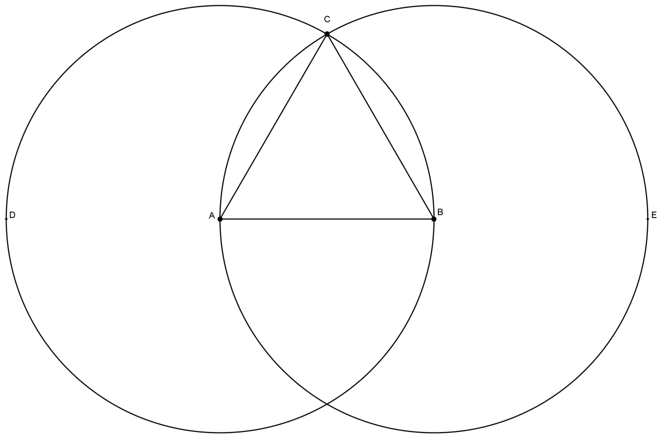

Fig. 6 | Diagramm

This is obvious from the diagram. However, intuitively, one would not consider the following statement to be a logical truth:

$\forall xyab$ (x is a circle around a through $b \wedge y$ is a circle around b through $a \rightarrow x$ and y intersect)

Nor does this sentence seem to follow from Euclid's definitions, postulates, and common notions.

Euclid defines a circle as a figure enclosed by a single line such that all lines from a certain privileged point – the center – to the enclosing line are equal (I, def. 15). The enclosing line is called the circumference (perhaps in def .15 – there is a textual difficulty – but certainly in def. 17). And the third postulate is "to describe a circle with any given center and radius". We will here use the word 'circle' for the enclosing line. Like straight lines and triangles, circles are defined by certain points, namely the center and another point that determines the straight line that is the radius. By analogy with the predicates 'S' and 'Δ', we introduce a three-place predicate, 'K'. '$Kxab$' is to be read 'x is a circle around center a through b'. By analogy with Postulate 1, Postulate 3 from the *Elements* can be stated:

(Ax 19) $\forall xys\,(Px \wedge Py \wedge Ssxy \rightarrow \exists z\, Kzxy)$

We base our definition of the predicate '*K*' on Euclid's definition 15:

Definition: $$Kxyz :\leftrightarrow Py \wedge Pz \wedge z \sqsubseteq x \wedge \forall w\,(Pw \wedge w \sqsubseteq x \rightarrow y \neq w) \\ \wedge \forall wss'\,(Pw \wedge Ssyz \wedge Ss'yw \rightarrow\ (w \sqsubseteq x \leftrightarrow s \equiv s'))$$

An immediate consequence of this definition is:

Lemma 2: $\vdash \forall xyzkst\,(Kkxy \wedge Ssxy \wedge Stxz \wedge z \sqsubseteq k \rightarrow s \equiv t)$

In order to fill the hole in the proof, it would be natural to use this axiom:

$$\forall abkk'\,(Pa \wedge Pb \wedge a \neq b \wedge Kkab \wedge Kk'ba \rightarrow \exists c\,(Pc \wedge c \sqsubseteq k \wedge c \sqsubseteq k'))$$

But this is not, in fact, sufficient. For the intersection points of the circles cannot be collinear with the centers of the circles. That is, Euclid relies on this fact in his proof, since otherwise the three points would not be vertices of a triangle. Since we do not see how to infer this from the axioms (those we have stated thus far), we strengthen this axiom for the moment in the following way:

(Ax 20⁻) $$\forall abkk' \, (Pa \wedge Pb \wedge a \neq b \wedge Kkab \wedge Kk'ba \rightarrow \exists c \, (Pc \wedge c \sqsubseteq k \wedge c \sqsubseteq k' \wedge \neg Coll(abc)))$$

This formulation asserts the existence of (at least) one point of intersection, whereas we will later need two points of intersection (e.g., in Proposition 10), which lie on opposite sides of the line between the circles' centers. We thus suggest the following, yet stronger axiom:

(Ax 20) $$\forall abkk's \, (Ssab \wedge Kkab \wedge Kk'ba \rightarrow \exists cc'td \, (Pc \wedge c \sqsubseteq k \wedge c \sqsubseteq k' \wedge Pc' \wedge c' \sqsubseteq k \wedge c' \sqsubseteq k' \wedge Stcc' \wedge Pd \wedge d \sqsubseteq s \wedge d \sqsubseteq t \wedge \neg Coll(abc) \wedge \neg Coll(abc') \wedge \neg Coll(cac') \wedge \neg Coll(cbc') \wedge \neg Coll(acd) \wedge \neg Coll(bcd)))$$

3.3 Proof of *Elements* I.1

The demonstration of I.1 does not assert the existence of a triangle until its concluding line. Up to that point, the demonstration leads up to the following intermediate conclusion:

$$\forall xab \, (Sxab \rightarrow \exists css' \, (\neg Coll(abc) \wedge Ssac \wedge Ss'bc \wedge x \equiv s \wedge x \equiv s'))$$

We will begin by proving this statement.

Proposition 1, Version 1: Let $\Sigma := \{$(Ax 1), (Ax 2), (Ax 3), (Ax 5), (Ax 14), (Ax 15), (Ax 19), (Ax 20⁻)$\}$; then

$$\Sigma \vdash \forall xab \, (Sxab \rightarrow \exists css' \, (\neg Coll(abc) \wedge Ssac \wedge Ss'bc \wedge x \equiv s \wedge x \equiv s'))$$

Proof: With (Ax 1), (Ax 2) and (Ax 19), we have:

$$\text{(Ax 1, 2, 19)} \vdash Sxab \rightarrow a \neq b \wedge Pa \wedge Pb \rightarrow a \neq b \wedge Pa \wedge Pb \wedge \exists k \, Kkab \wedge \exists k \, Kkba$$

With (Ax 20⁻), it then follows that:

$$\text{(Ax 1, 2, 19, 20}^-\text{)} \vdash Sxab \rightarrow \exists ckk' (Pc \wedge c \sqsubseteq k \wedge c \sqsubseteq k' \wedge Kkab \wedge Kk'ba \wedge \neg Col(abc))$$

By the definition of 'K', this entails:

$$\text{(Ax 1, 2, 19, 20}^-\text{)} \vdash Sxab \rightarrow \exists ckk' \, (\neg Coll(abc) \wedge Pc \wedge c \sqsubseteq k \wedge \forall wss' \, (Pw \wedge w \sqsubseteq k \wedge Ssab \wedge Ss'aw \rightarrow s \equiv s') \wedge c \sqsubseteq k' \wedge \forall wss' \, (Pw \wedge w \sqsubseteq k' \wedge Ssba \wedge Ss'bw \rightarrow s \equiv s'))$$

Therefore,

$$\text{(Ax 1, 2, 19, 20}^-\text{)} \vdash Sxab \rightarrow \exists ckk' \, (\neg Coll(abc) \wedge Pc \wedge c \sqsubseteq k \wedge \forall s' \, (Pc \wedge c \sqsubseteq k \wedge Sxab \wedge Ss'ac \rightarrow x \equiv s') \wedge Pc \wedge c \sqsubseteq k' \wedge \forall s' \, (Pc \wedge c \sqsubseteq k' \wedge Sxba \wedge Ss'bc \rightarrow x \equiv s'))$$

From (Ax 1) and (Ax 3), we then get:

$$\text{(Ax 1, 2, 3, 19, 20}^-\text{)} \vdash Sxab \rightarrow \exists c (\neg Coll(abc) \wedge Pa \wedge Pb \wedge Pc \wedge \forall s' \, (Ss'ac \rightarrow x \equiv s') \wedge Pc \wedge \forall s' \, (Ss'bc \rightarrow x \equiv s'))$$

From this and Lemma 1, we can infer the claim to be demonstrated:

$$\Sigma \vdash Sxab$$
$$\to \exists c(\neg Coll(abc) \wedge \exists s\, Ssac \wedge \forall s' (Ss'ac \to x \equiv s') \wedge \exists s\, Ssbc \wedge \forall s' (Ss'bc \to x \equiv s'))$$
$$\to \exists css' (\neg Coll(abc) \wedge Ssac \wedge Ss'bc \wedge x \equiv s \wedge x \equiv s')$$

From the conclusion of Proposition 1, Version 1, and (Ax 18) (plus (Ax 1)), we can immediately infer:

Proposition 1, Version 2: Take Σ from Proposition 1, Version 1. Then

$$\Sigma \cup \{\text{Ax 18}\} \vdash \forall xab\, (Sxab \to \exists ycss' (\Delta yabc \wedge Ssac \wedge Ss'bc \wedge x \equiv s \wedge x \equiv s'))$$

3.4 Proofs of *Elements* I.2–3

For the next two propositions, we will also need to fill in gaps, but this is yet more complicated than for Proposition 1. Propositions 2 and 3 are problems. Reformulated as assertions, they are:

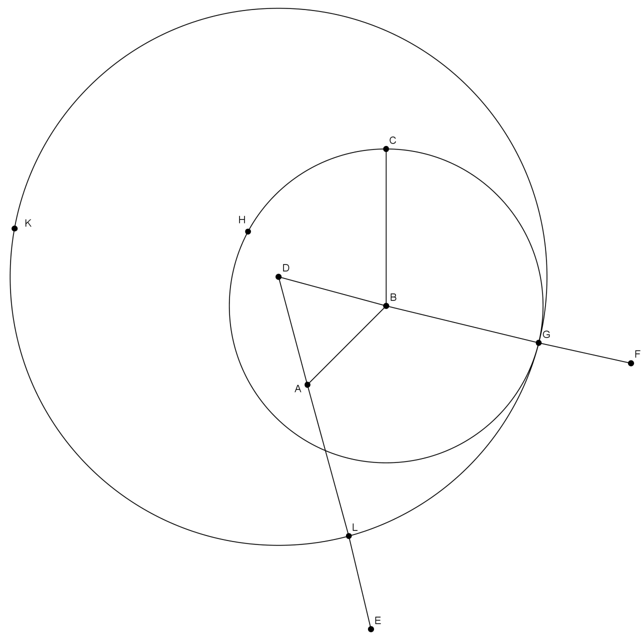

Fig. 7 | Diagramm

Proposition 2. "For any given point and given straight line, there is a straight line equal to the given straight line with the given point as an endpoint." (see Fig. 7).

Proposition 3. "For any two given unequal straight lines, there is a straight line that is equal to the lesser given straight line and is part of the greater given straight line." (see Fig. 8).

It is clear that Proposition 2 plays a crucial role in the demonstration of Proposition 3. But Proposition 3 is more than a corollary to Proposition 2. For Proposition 3 is the first occasion that Euclid speaks of straight lines being larger than others. That requires new axioms.

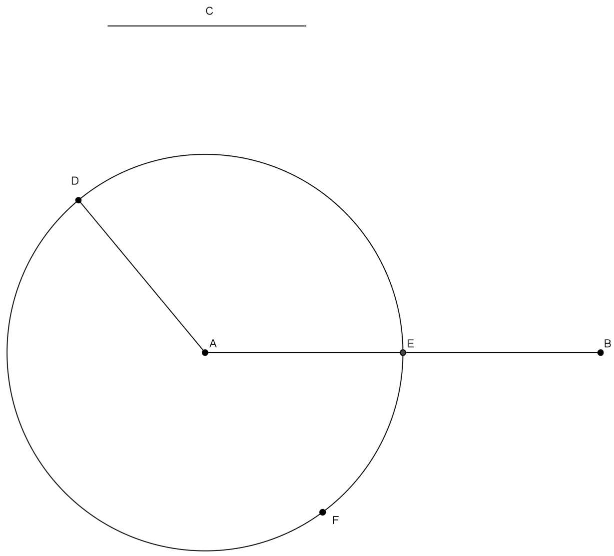

Fig. 8 | Diagramm

However, there are several significant gaps in the demonstration of Proposition 2. For instance, how do we know that there is a point G on the circle around B, which lies on the straight line *BG*? See Figure 7. In theproposition demonstration, Postulate 2 is used to justify this claim. Postulate 2 is 'to extend a given straight line continuously in a straight line.' But it is not clear how to formulate Postulate 2 in our framework.

In order to demonstrate Proposition 2, we propose three new axioms, (Ax 21.1), (Ax 21.2), and (Ax 22).

(Ax 21.1) $\forall xyzsk\,(Ssxy \wedge Kkxz \rightarrow \exists ws'\,(Pw \wedge Ss'yw \wedge s \sqsubseteq s' \wedge w \sqsubseteq k))$

(Ax 21.2) $\forall xyzsk\,(Ssxy \wedge Kkxz \rightarrow \exists wtt'\,(Pw \wedge Stxw \wedge St' \wedge t' \sqsubseteq s \wedge t' \sqsubseteq t \wedge w \sqsubseteq k))$

On the one hand, these axioms are both visually plausible. On the other hand, they are, admittedly, formulated to fill the gap in this demonstration. We will have to critically examine whether they are merely *ad hoc* and useful only in this case, or whether they are generally useful. From (Ax 21.1) and (Ax 21.2), we first derive this lemma:

Lemma 3: (Ax 5, 21.1) $\vdash \forall xyzsk\,(Ssxy \wedge Kkxz \rightarrow \exists ws't\,(Pw \wedge Ss'yw \wedge s \sqsubseteq s' \wedge w \sqsubseteq k \wedge Stxw))$
Proof: With (Ax 21.1) in the first step, the definition of 'K' in the second step, and (Ax 5) in the third step, we have:

$$\begin{aligned}(\text{Ax } 5, 21.1) \vdash Ssxy \wedge Kkxz &\rightarrow \exists ws'\,(Pw \wedge Ss'yw \wedge s \sqsubseteq s' \wedge w \sqsubseteq k \wedge Kkxz)\\ &\rightarrow \exists ws'\,(Pw \wedge Ss'yw \wedge s \sqsubseteq s' \wedge w \sqsubseteq k \wedge x \neq w)\\ &\rightarrow \exists ws'\,(Pw \wedge Ss'yw \wedge s \sqsubseteq s' \wedge w \sqsubseteq k \wedge \exists t\, Stxw)\end{aligned}$$

Moreover, the so-called Common Notions play a nontrivial role for the first time in the *Elements*. In particular, we need an axiom that corresponds to Common Notion 3 ('if equals be taken away from equals, the remainders are equal'):

(Ax 22) $\forall xyzx'y'z'ss'tt'vv' \, (Ssxy \wedge Ss'x'y' \wedge s \equiv s' \wedge Stxz \wedge St'x'z' \wedge t \equiv t' \wedge t \sqsubseteq s \wedge t' \sqsubseteq s' \wedge Svzy \wedge Sv'z'y' \rightarrow v \equiv v')$

It turns out that a variant on (Ax 21.2) is useful for the proof of Proposition 2, formulated not in terms of the part-whole relation but in terms of the greater-less relation. Thus we first introduce the 'greater-less' predicate for straight lines and gives axioms for it. This figures in the proof of Proposition 2. And Proposition 3 will then be proved using both these axioms and Proposition 2.

Let us extend L[E] by the two-place predicate '$<$'. '$x < y$' is read '(straight line) y is longer than (straight line) x'. The following should be plausible axioms for '$<$':

(Ax 23) $\forall st \, (s < t \rightarrow Ss \wedge St)$

(Ax 24) $\forall stu \, (s < t \wedge t < u \rightarrow s < u)$

(Ax 25) $\forall st \, (Ss \wedge St \rightarrow s < t \vee s \equiv t \vee t < s)$

(Ax 26) $\forall st \, (s \equiv t \rightarrow \neg(s < t))$

(Ax 27) $\forall st \, (Ss \wedge St \wedge s \sqsubseteq t \rightarrow s \leq t)$[9]

(Ax 28) $\forall stabc \, (Ssab \wedge Stac \wedge \exists t' \, (St' \wedge t' \sqsubseteq s \wedge t' \sqsubseteq t) \wedge s \leq t \rightarrow s \sqsubseteq t)$

where we have defined: $s \leq t :\leftrightarrow s < t \vee s \equiv t$

Lemma 4: (i) $\vdash \forall st \, (s \equiv t \rightarrow s \leq t)$
(ii) (Ax 8, 26) $\vdash \forall s \, (Ss \rightarrow \neg s < s)$
(iii) (Ax 9, 26) $\vdash \forall st \, (s \equiv t \rightarrow \neg(t < s))$
(iv) (Ax 9, 23, 24, 25, 26) $\vdash \forall stu \, (s \equiv t \wedge s < u \rightarrow t < u)$
(v) (Ax 9, 23, 24, 25, 26) $\vdash \forall stu \, (s \equiv t \wedge u < s \rightarrow u < t)$
(vi) (Ax 9, 10, 23, 24, 25, 26) $\vdash \forall stu \, (s \leq t \wedge t \leq u \rightarrow s \leq u)$

Proof: (i) Immediate from the definition.
(ii) (Ax 26) implies '$\forall s \, (s \equiv s \rightarrow \neg(s < s))$'. The claim follows by (Ax 8).
(iii) Follows from (Ax 9) and (Ax 26).
(iv) By (Ax 23) and (Ax 25), we have

(*) (Ax 23, 25) $\vdash s \equiv t \wedge s < u \wedge \neg t < u \rightarrow St \wedge Su \wedge s < u \wedge (t \equiv u \vee u < t)$

Yet from (Ax 9) and (Ax 26), we have

(**) (Ax 9, 26) $\vdash s \equiv t \wedge s < u \wedge \neg t < u \wedge t \equiv u \rightarrow s < u \wedge s \equiv u \rightarrow \bot$

While from (Ax 24) and (Ax 26), we have

(***) (Ax 24, 26) $\vdash s \equiv t \wedge s < u \wedge \neg t < u \wedge u < t \rightarrow s < t \wedge s \equiv t \rightarrow \bot$

9 This corresponds to Common Notion 5, 'The whole is greater than the part.'

Together (*), (**), and (***) entail:

$$(\text{Ax } 9, 23, 24, 25, 26) \vdash s \equiv t \wedge s < u \wedge \neg t < u \rightarrow \bot,$$

which yields the claim to be proved.

(v) Proved like (iv).

(vi) Follows from (Ax 10) and (Ax 24) and (iv) or (v).

Lemma 5: (i) (Ax 11, 28) $\vdash \forall xyzst\, (Ssxy \wedge Stxz \wedge t \sqsubseteq s \wedge s \leq t \rightarrow s \sqsubseteq t)$

(ii) (Ax 11, 13, 28) $\vdash \forall xyzst\, (Ssxy \wedge Stxz \wedge t \sqsubseteq s \wedge s \leq t \rightarrow s = t)$

(iii) (Ax 5, 11, 13, 14, 15, 16, 28) $\vdash \forall xyzst\, (Ssxy \wedge Stxz \wedge z \sqsubseteq s \wedge s \equiv t \rightarrow s = t \wedge y = z)$

Proof: (i) With (Ax 11) for the first step and (Ax 28) for the second step, we have

$$(\text{Ax } 11, 28) \vdash Ssxy \wedge Stxz \wedge t \sqsubseteq s \wedge s \leq t \rightarrow Ssxy \wedge Stxz \wedge \exists t'\, (St' \wedge t' \sqsubseteq s \wedge t' \sqsubseteq t) \wedge s \leq t$$
$$\rightarrow s \sqsubseteq t$$

(ii) From (i), by (Ax 13).

(iii) By (Ax 15) and then by (Ax 16), we have:

$$(\text{Ax } 15, 16) \vdash Ssxy \wedge z \sqsubseteq s \wedge Stxz \rightarrow Ssxy \wedge x \sqsubseteq s \wedge z \sqsubseteq s \wedge Stxz$$
$$\rightarrow t \sqsubseteq s$$

Together with (ii), this entails:

$$(\text{Ax } 11, 13, 15, 16, 28) \vdash Ssxy \wedge z \sqsubseteq s \wedge Stxz \wedge s \equiv t \rightarrow Ssxy \wedge Stxz \wedge t \sqsubseteq s \wedge s \leq t$$
$$\rightarrow s = t$$

This and Lemma 1(iii) imply then:

$$(\text{Ax } 5, 11, 13, 14, 15, 16, 28) \vdash Ssxy \wedge z \sqsubseteq s \wedge Stxz \wedge s \equiv t \rightarrow s = t \wedge Ssxy \wedge Stxz$$
$$\rightarrow y = z,$$

and the claim is thereby proven.

With these preliminary axioms (Ax 1) – (Ax 28 and the new '$<$' predicate, we are ready to prove Propositions 2 and 3.

We divide Proposition 2 into two cases. In the first case, the endpoint at which the constructed line is to be place is identical with an endpoint of the given line. In the second case, neither endpoint of the given line is identical with the given point. The proof of the first case is straightforward. The proof of the second case is more involved.

Proposition 2, 1st case: (Ax 8) $\vdash \forall abcu\, (a = b \wedge Pa \wedge Pb \wedge Pc \wedge Subc \rightarrow \exists lw\, (Pl \wedge Swal \wedge u \equiv w))$

Proof: Using (Ax 8) in the second step, we have

$$(\text{Ax } 8) \vdash a = b \wedge Pa \wedge Pb \wedge Pc \wedge Subc \rightarrow Pc \wedge Suac$$
$$\rightarrow Pc \wedge Suac \wedge u \equiv u$$
$$\rightarrow \exists lw\, (Pl \wedge Swal \wedge u \equiv w)$$

In the remaining section, "$\vdash$" means "(Ax 1) – (Ax 28) $\vdash$".

Lemma 6: (i) $\vdash Sxab \wedge Pa \wedge Pb \wedge Pc \wedge Subc$
$\to \exists dyzk\, (Pd \wedge \neg Coll(abd) \wedge Syad \wedge Szbd \wedge x \equiv y \wedge x \equiv z \wedge\ Kkbc)$
(ii) $\vdash Subc \wedge Szbd \wedge Kkbc$
$\to \exists gs't'\, (Pg \wedge Ss'dg \wedge z \sqsubseteq s' \wedge g \sqsubseteq k \wedge St'bg \wedge u \equiv t' \wedge\ t' \sqsubseteq s')$
(iii) $\vdash z \equiv y \wedge z \sqsubseteq s' \wedge Syda \wedge Ss'dg$
$\to \exists k'lt\, (Kk'dg \wedge Pl \wedge Stdl \wedge y \sqsubseteq t \wedge l \sqsubseteq k' \wedge\ s' \equiv t)$
(iv) $\vdash z \equiv y \wedge Stdl \wedge Syda \wedge s' \equiv t \wedge Ss'dg \wedge Szdb \wedge St'bg \wedge z \sqsubseteq s' \to a \neq l$

Proof: (i) From the first version of the formalization of Proposition 1, we have:

$$\vdash Pa \wedge Pb \wedge Sxab \to \exists dyz\, (Pd \wedge \neg Coll(abd) \wedge Syad \wedge Szbd \wedge x \equiv y \wedge x \equiv z)$$

Also, by (Ax 19),

$$\vdash Pb \wedge Pc \wedge Subc \to \exists k\, Kkbc$$

These statements imply the claim to be proved.

(ii) From Lemma 3, we have

$$\vdash Szbd \wedge Kkbc \to \exists gs't'\, (Pg \wedge Ss'dg \wedge z \sqsubseteq s' \wedge g \sqsubseteq k \wedge St'bg)$$

From this, using (Ax 15) in the first step, (Ax 12) in the second step, and (Ax 16) in the third step, we obtain:

$$\begin{aligned}\vdash Szbd \wedge Kkbc &\to \exists gs't'\, (Pg \wedge Ss'dg \wedge z \sqsubseteq s' \wedge g \sqsubseteq k \wedge St'bg \wedge b \sqsubseteq z \wedge g \sqsubseteq s')\\ &\to \exists gs't'\, (Pg \wedge Ss'dg \wedge z \sqsubseteq s' \wedge g \sqsubseteq k \wedge St'bg \wedge b \sqsubseteq s' \wedge g \sqsubseteq s')\\ &\to \exists gs't'\, (Pg \wedge Ss'dg \wedge z \sqsubseteq s' \wedge g \sqsubseteq k \wedge St'bg \wedge t' \sqsubseteq s')\end{aligned}$$

Therefore

(*) $\vdash Subc \wedge Szbd \wedge Kkbc \to \exists gs't' (Pg \wedge Ss'dg \wedge z \sqsubseteq s' \wedge Kkbc \wedge Subc \wedge St'bg \wedge g \sqsubseteq k \wedge t' \sqsubseteq s')$

Now, by Lemma 2,

(**) $\vdash Kkbc \wedge Subc \wedge St'bg \wedge g \sqsubseteq k \to u \equiv t'$

(*) and (**) entail the claim to be proved.

(iii) (Ax 21.2) yieds

$$\vdash Syda \wedge Kk'dg \to \exists ltt'\, (Pl \wedge Stdl \wedge St' \wedge t' \sqsubseteq y \wedge t' \sqsubseteq t \wedge l \sqsubseteq k')$$

which entails

$$\vdash Syda \wedge Ss'dg \wedge Kk'dg \to \exists ltt'\, (Pl \wedge St' \wedge t' \sqsubseteq y \wedge t' \sqsubseteq t \wedge Kk'dg \wedge Ss'dg \wedge Stdl \wedge l \sqsubseteq k')$$

In addition, by Lemma 2,

$\vdash Kk'dg \wedge Ss'dg \wedge Stdl \wedge l \sqsubseteq k' \rightarrow s' \equiv t$

These two lines imply

$\vdash Syda \wedge Ss'dg \wedge Kk'dg \rightarrow \exists ltt'(Pl \wedge Syda \wedge Stdl \wedge St' \wedge t' \sqsubseteq y \wedge t' \sqsubseteq t \wedge l \sqsubseteq k' \wedge s' \equiv t)$

Now this implies, with (Ax 27) in the first step, Lemma 4 in the second step and (Ax 28) in the third step:

$$
\begin{aligned}
&\vdash z \equiv y \wedge z \sqsubseteq s' \wedge Syda \wedge Ss'dg \wedge Kk'dg \\
&\quad \rightarrow \exists ltt'(Pl \wedge Syda \wedge Stdl \wedge St' \wedge t' \sqsubseteq y \wedge t' \sqsubseteq t \wedge l \sqsubseteq k' \wedge s' \equiv t \wedge z \equiv y \wedge z \leq s') \\
&\quad \rightarrow \exists ltt'\,(Pl \wedge Syda \wedge Stdl \wedge St' \wedge t' \sqsubseteq y \wedge t' \sqsubseteq t \wedge y \leq t \wedge l \sqsubseteq k' \wedge s' \equiv t) \\
&\quad \rightarrow \exists ltt'\,(Pl \wedge Syda \wedge Stdl \wedge St' \wedge t' \sqsubseteq y \wedge t' \sqsubseteq t \wedge y \sqsubseteq t \wedge l \sqsubseteq k' \wedge s' \equiv t) \\
&\quad \rightarrow \exists lt\,(Pl \wedge Stdl \wedge y \sqsubseteq t \wedge l \sqsubseteq k' \wedge s' \equiv t)
\end{aligned}
$$

By (Ax 19), the claim follows.

(iv) By (Ax 4) in the first step and (Ax 10) in the second step, we have:

(*) $\vdash z \equiv y \wedge Stdl \wedge Syda \wedge s' \equiv t \wedge a = l \rightarrow z \equiv y \wedge y = t \wedge s' \equiv t$
$\rightarrow z \equiv s'$

In addition, with (Ax 15) in the first step and (Ax 12) in the second step, we have:

(**) $\vdash Ss'dg \wedge z \sqsubseteq s' \wedge Szdb \wedge z \equiv s' \rightarrow Ss'dg \wedge z \sqsubseteq s' \wedge b \sqsubseteq z \wedge z \equiv s'$
$\rightarrow Ss'dg \wedge b \sqsubseteq s' \wedge Szdb \wedge z \equiv s'$

From this, by Lemma 5(iii), we get:

$\vdash Ss'dg \wedge z \sqsubseteq s' \wedge Szdb \wedge z \equiv s' \rightarrow b = g$

(*) and (**) yield

(+) $\vdash z \equiv y \wedge Stdl \wedge Syda \wedge s' \equiv t \wedge Ss'dg \wedge Szdb \wedge z \sqsubseteq s' \wedge a = l \rightarrow b = g$

But by (Ax 2), we also have:

(++) $\vdash St'bg \rightarrow b \neq g$

(+) and (++) imply the claim to be proved.

With this additional lemma now proved, we can go on to complete our proof of Proposition 2.

Proposition 2, 2nd case: $\vdash \forall abcu(a \neq b \wedge Pa \wedge Pb \wedge Pc \wedge Subc \rightarrow \exists lw(Pl \wedge Swal \wedge u \equiv w))$
Proof: Let α be the formula

$Pa \wedge Pd \wedge \neg Coll(abd) \wedge Syad \wedge Szbd \wedge x \equiv y \wedge x \equiv z \wedge Kkbc \wedge Pg \wedge Ss'dg \wedge z \sqsubseteq s'$
$\wedge\, g \sqsubseteq k \wedge St'bg \wedge u \equiv t' \wedge t' \sqsubseteq s' \wedge z \equiv y \wedge Kk'dg \wedge Pl \wedge Stdl \wedge y \sqsubseteq t \wedge l \sqsubseteq k' \wedge s' \equiv t \wedge a \neq l$

By Lemma 6(i) and 6(ii), we have:

$$\vdash Sxab \wedge Pa \wedge Pb \wedge Pc \wedge Subc \rightarrow \exists dyzk\,(Pd \wedge \neg Coll(abd) \wedge Syad \wedge Szbd \wedge x \equiv y \wedge x \equiv z \wedge Kkbc \wedge \exists gs't'\,(Pg \wedge Ss'dg \wedge z \sqsubseteq s' \wedge g \sqsubseteq k \wedge St'bg \wedge u \equiv t' \wedge t' \sqsubseteq s'))$$

Together with (Ax 3), (Ax 9), and (Ax 10), this implies:

$$\vdash Sxab \wedge Pa \wedge Pb \wedge Pc \wedge Subc \rightarrow \exists dyzk \exists gs't' (Pd \wedge \neg Coll(abd) \wedge Syad \wedge Szbd \wedge x \equiv y \wedge x \equiv z \wedge Kkbc \wedge Pg \wedge Ss'dg \wedge z \sqsubseteq s' \wedge g \sqsubseteq k \wedge St'bg \wedge u \equiv t' \wedge t' \sqsubseteq s' \wedge z \equiv y)$$

Therefore, by Lemma 6(iii) and 6(iv):

$$(+) \quad \vdash Sxab \wedge Pa \wedge Pb \wedge Pc \wedge Subc \rightarrow \exists dyzk\, \exists gs't'\, (Pd \wedge \neg Coll(abd) \wedge Syad \wedge Szbd \wedge x \equiv y \wedge x \equiv z \wedge Kkbc \wedge Pg \wedge Ss'dg \wedge z \sqsubseteq s' \wedge g \sqsubseteq k \wedge St'bg \wedge u \equiv t' \wedge t' \sqsubseteq s' \wedge z \equiv y \wedge \exists k'lt\,(Kk'dg \wedge Pl \wedge Stdl \wedge y \sqsubseteq t \wedge l \sqsubseteq k' \wedge s' \equiv t \wedge a \neq l))) \rightarrow \exists dyzkgs't'k'lt\, \alpha$$

From (Ax 5), we have:

$$(*) \quad \vdash \alpha \rightarrow \exists w\,(Swal \wedge Pl \wedge Stdl \wedge Ss'dg \wedge s' \equiv t \wedge Syda \wedge Szdb \wedge y \sqsubseteq t \wedge z \sqsubseteq s' \wedge y \equiv z \wedge St'bg \wedge u \equiv t')$$

Furthermore, from (Ax 22), we have:

$$(**) \quad \vdash Stdl \wedge Ss'dg \wedge s' \equiv t \wedge Syda \wedge Szdb \wedge y \sqsubseteq t \wedge z \sqsubseteq s' \wedge y \equiv z \wedge Swal \wedge St'bg \rightarrow\; w \equiv t'$$

(*) and (**) entail, together with (Ax 9) and (Ax 10) in the second step:

$$\vdash \alpha \rightarrow \exists w\,(Pl \wedge Swal \wedge w \equiv t' \wedge u \equiv t') \rightarrow \exists lw\,(Pl \wedge Swal \wedge u \equiv w)$$

This and (+) imply

$$\vdash \exists x\, Sxab \wedge Pa \wedge Pb \wedge Pc \wedge Subc \rightarrow \exists lw\,(Pl \wedge Swal \wedge u \equiv w)$$

With (Ax 5), the claim to be proven follows.

Without further ado, we turn to Proposition 3.

Proposition 3: $\vdash \forall ss'\,(Ss \wedge Ss' \wedge s' < s \rightarrow \exists t\,(St \wedge t \sqsubseteq s \wedge s' \equiv t))$
Proof: Let β be the formula

$$Sxad \wedge x \equiv s' \wedge Kkad \wedge Pw \wedge Staw \wedge St' \wedge t' \sqsubseteq t \wedge t' \sqsubseteq s \wedge w \sqsubseteq k \wedge Ssab \wedge s' < s$$

By Proposition 2 and (Ax 1), we have:

$$\vdash Ssab \wedge Ss'uv \wedge s' < s \rightarrow \exists dx\,(Sxad \wedge x \equiv s')$$

From this, it follows by (Ax 19)

(*) $\vdash Ssab \wedge Ss' \wedge s' < s \rightarrow \exists dxk\,(Sxad \wedge x \equiv s' \wedge Kkad)$

Moreover, by (Ax 21.2)

(**) $\vdash Ssab \wedge Kkad \rightarrow \exists wtt'\,(Pw \wedge Staw \wedge St' \wedge t' \sqsubseteq t \wedge t' \sqsubseteq s \wedge w \sqsubseteq k)$

From (*) and (**) we have:

(+) $\vdash Ssab \wedge Ss' \wedge s' < s \rightarrow \exists t \exists dxkwt'\, \beta$

From Lemma 2, we have:

$\vdash Kkad \wedge Sxad \wedge Staw \wedge w \sqsubseteq k \rightarrow x \equiv t$

and also, by (Ax 9) and (Ax 10),

$\vdash Kkad \wedge Sxad \wedge Staw \wedge w \sqsubseteq k \wedge x \equiv s' \rightarrow s' \equiv t$

Thus

(++) $\vdash \beta \rightarrow s' \equiv t$

Furthermore, using Lemma 4 in the first step and (Ax 28) in the second,

$$\begin{aligned}\vdash Ssab \wedge Staw \wedge St' \wedge t' \sqsubseteq t \wedge t' \sqsubseteq s \wedge t \equiv s' \wedge s' < s\\ \rightarrow Ssab \wedge Staw \wedge St' \wedge t' \sqsubseteq t \wedge t' \sqsubseteq s \wedge t \leq s\\ \rightarrow t \sqsubseteq s\end{aligned}$$

Hence

(+++) $\vdash \beta \rightarrow t \sqsubseteq s$

(+), (++) and (+++) imply the claim to be proven.

3.5 Remark on propositions I.4–10

We have also worked out proofs of Propositions 4 through 6 and 8 through 10.[10] Or rather, we have (following Hilbert and many others) adopted Propositions 4 and 8 as axioms, and worked out proofs of the others. Proposition 4 says, roughly, that if two triangles have two pairs of equal sides and the angle between the equal sides equal, then they are congruent. Proposition 8 says that if two triangles have three pairs of equal sides, then the angles contained by the pairs of equal sides are equal.

Proposition 5, the famous *pons asinorum*, whose first proof is (somewhat implausibly) attributed to Thales, says that isosceles triangles have their base angles (and exterior base angles) equal. Proposition 6 states its converse.

Proposition 9 is "to cut a given straight angle in half". Proposition 10 is "to cut a given bounded straight line in half". We have formalized them, like Propositions 1 through 3, as $\forall\exists$-sentences.

10 Proposition 7 is difficult to formalize and remains to be dealt with. One importance source of the difficulty is the predicate '[point] …is on the other side of [line] ..'

For reasons of space, we cannot include our presentation of these proofs here, but, to give the reader at least a rough sense of how we have proceeded, we describe briefly here the predicates and axioms we introduced.

The propositions in question (starting already with Proposition 4) use the concept of an angle and they speak of angles being equal to or less than one another. The treatment of angles has proved particularly challenging. We have extended L[E] by corresponding six-place predicates, ‘$bc_a \tilde{\equiv}_d ef$’ (read ‘the angle at point a determined by segments ab and ac is equal to the angle at point d determined by line segments de and df’’) and ‘$bc_a \tilde{<}_d ef$’ (read ‘the angle at point a determined by segments ab and ac is less than or equal to the angle at point d determined by line segments de and df’’).

These predicates are governed by the following axioms:

(Ax 29) $\forall abca'b'c' \, (bc_a \tilde{\equiv}_{a'} b'c' \rightarrow Pa \wedge Pb \wedge Pc \wedge Pa' \wedge Pb' \wedge Pc')$

(Ax 30) $\forall abc \, (Pa \wedge Pb \wedge Pc \rightarrow bc_a \tilde{\equiv}_a bc)$

(Ax 31) $\forall abc \, (Pa \wedge Pb \wedge Pc \rightarrow bc_a \tilde{\equiv}_a cb)$

(Ax 32) $\forall abca'b'c'a''b''c'' \, (bc_a \tilde{\equiv}_{a'} b'c' \wedge b'c'_{a'} \tilde{\equiv}_{a''} b''c'' \rightarrow bc_a \tilde{\equiv}_{a''} b''c'')$

(Ax 33) $\forall abca'b'c' \, (bc_a \tilde{<}_{a'} b'c' \rightarrow Pa \wedge Pb \wedge Pc \wedge Pa' \wedge Pb' \wedge Pc')$

(Ax 34) $\forall abcdefss'tt'uu' \, (\neg Coll(abc) \wedge \neg Coll(def) \wedge Ssab \wedge Ss'de \wedge s \equiv s'$
$\wedge \, Stac \wedge St'df \wedge t \equiv t' \wedge bc_a \tilde{\equiv}_d ef \wedge Subc \wedge Su'ef \rightarrow u \equiv u')$

4 Systematic summary and refinement of the logical system

We have formulated a first-order language L[E] and developed in it a certain *Theory of Euclidean Geometry*, EG. We have derived some propositions of this theory. Thus far, the vocabulary of this language contains only the following non-logical signs:

the one-place predicate:	‘P’
the two-place predicates:	‘$\equiv$’, ‘$\sqsubseteq$’
the three-place predicate:	‘S’
the four-place predicate:	‘Δ’
the six-place predicates:	‘$\tilde{\equiv}$’, ‘$\tilde{<}$’

Alongside the usual axioms and rules of classical first-order predicate logic with identity, we stipulated 36 axioms. We have here stated and proved Propositions 1 through 3. In other work, we have also stated propositions 4 through 6 and 8 through 10, assuming propositions 4 and 8 as axioms and proving the others.

This shows that L[E] and EG are sufficient for a detailed presentation of the *Elements* through Proposition 10 (except for Proposition 7). But it will be obvious to any reader of the *Elements* that further axioms are necessary for later parts of the work (e.g., the theory of ratio in Book V, the theory of number in Books VII through IX, the three-dimensional geometry of Book XI). It is to be expected that more axioms will be needed even for the remainder of Book I and for the other elementary books on plane geometry without ratios (II through IV). Both L[E] and EG are here *preliminary*. Both the language and the theory will need to be checked and perhaps altered.

At the same time, it is conspicuous how many axioms are already necessary for EG. Moreover, some of them are fairly complicated formulas. Could we not simplify somewhat? This can be broken down into three more specific question. (1) Can some axioms be eliminated by deriving them from others? (2) Can some primitive predicates be defined in terms of others? (3) Could we replace some axioms with other, preferable axioms? We now go on to address these questions in two subsections.

4.1 Elimination of axioms by derivation

Are there axioms that we can eliminate by deriving them from others? – The answer is yes. Here is an example, which might be somewhat surprising: (Ax 3) and (Ax 4) are derivable.

Lemma 7: (i) (Ax 13, 15, 16) $\vdash$ (Ax 4)
(ii) (Ax 1, 2, 5, 13, 15, 16) $\vdash$ (Ax 3)
Proof: (i) From (Ax 16), we have:

$$\text{(Ax 16)} \vdash Ssxy \wedge Ss'xy \wedge x \sqsubseteq s \wedge y \sqsubseteq s \to s' \sqsubseteq s$$

From this follows, by (Ax 15):

$$\text{(Ax 15, 16)} \vdash Ssxy \wedge Ss'xy \to s' \sqsubseteq s$$

By interchanging s and s', we also have:

$$\text{(Ax 15, 16)} \vdash Ss'xy \wedge Ssxy \to s \sqsubseteq s'$$

The claim to be proven follows from these two sentences and (Ax 13).

(ii) Analogously to the proof of (i), we can show:

$$\text{(Ax 13, 15, 16)} \vdash Ssyx \wedge Ss'xy \to s' = s$$

But this is equivalent to:

$$(*) \quad \text{(Ax 13, 15, 16)} \vdash Ssyx \to \forall s' \, (Ss'xy \to s' = s)$$

From (Ax 1), (Ax 2), and (Ax 5), we get:

$$(**) \quad \text{(Ax 1, 2, 5)} \vdash Ssyx \to Px \wedge Py \wedge x \neq y \\ \to \exists t \, Stxy$$

(*) and (**) imply:

$$\text{(Ax 1, 2, 5, 13, 15, 16)} \vdash Ssyx \to \exists t \, (Stxy \wedge t = s) \\ \to Ssxy$$

4.2 Elimination of axioms by new axioms and definitions

Is it possible to define some of our primitive predicates in terms of others? In what follows, we will consider two examples: a definition of 'Δ' and a definition of 'K' in terms of our other primitive predicates. These definitions not only fulfill their job from a proof-theoretic perspective but are also plausible in light of the intended reading of 'Δ' and 'K.' We use them in the final sections of our paper.

We will first consider the prospects for a definition of 'Δ.' To this end, let us consider the language L[E$^-$], which is L[E] without the predicates 'Δ' and 'K.'

In order to use the following mereological definition of 'Δ,' we will need to strengthen the mereological axioms of EG. To begin with, we will define points, just as the first definition of the *Elements* does, as objects having no (proper) parts.

Definition: $Px :\leftrightarrow \forall y\,(y \sqsubseteq x \rightarrow y = x)$

For another thing, we will require a *fusion scheme* for the language L[E$^-$].

(Ax FUS) Let ψ be a formula of L[E$^-$]; then the universal closure of the following formula is an axiom (an instance of the fusion scheme):

$$\exists x\,\psi \rightarrow \exists y \forall u\,(Pu \rightarrow (u \sqsubseteq y \leftrightarrow \exists x\,(\psi \wedge u \sqsubseteq x)))$$

The intuitive basis for the definition of 'Δ' is this: if a triangle T is determined by three non-collinear points, A, B, and C, and AB and AC are sides of T, and the point X is on AB and the point Y is on AC, then X and Y are parts of T and everything between X and Y is also part of T. Conversely, if T is a triangle determined by the non-collinear points A, B, and C, and P is a point that is part of T, then there is a straight line through P that intersects two sides of T.

Thus we define 'Δ' as follows:

Definitions:

$$Zuab :\leftrightarrow Pu \wedge \exists s\,(Ssab \wedge u \sqsubseteq s)$$
$$\Delta yabc :\leftrightarrow Pa \wedge Pb \wedge Pc \wedge \neg Coll(abc) \wedge$$
$$\forall u\,(Pu \rightarrow (u \sqsubseteq y \leftrightarrow \exists xvw\,(Zvab \wedge Zwac \wedge Sxvw \wedge u \sqsubseteq x)))$$

It is obvious that these *definientia* are expressed in L[E$^-$]. Using these definitions, the next lemma can be shown:

Lemma 8: (i) $\vdash$ (Ax 17)

(ii) (Ax 5, 14, 15, FUS) $\vdash$ (Ax 18)

Proof: (i) By definition.

(ii) Let ψ be the L[E$^-$] formula

$$\exists vw\,(Zvab \wedge Zwac \wedge Sxvw).$$

By Lemma 1 and (Ax 15) in the first step, we have:

$$\begin{aligned}(\text{Ax } 5, 14, 15) \vdash\ & Pa \wedge Pb \wedge Pc \wedge \neg Coll(abc) \\ & \rightarrow Pa \wedge \exists s\,(Ssab \wedge a \sqsubseteq s) \wedge Pc \wedge \exists s'\,(Ss'ac \wedge c \sqsubseteq s') \wedge \exists x\, Sxac \\ & \rightarrow \exists x\,(Zaab \wedge Zcac \wedge Sxac) \\ & \rightarrow \exists x \exists vw\,(Zvab \wedge Zwac \wedge Sxvw) \\ & \rightarrow \exists x\,\psi\end{aligned}$$

By (Ax FUS), it then follows that

$$\begin{aligned}& Pa \wedge Pb \wedge Pc \wedge \neg Coll(abc) \\ & \quad \rightarrow \exists y \forall u\,(Pu \rightarrow (u \sqsubseteq y \leftrightarrow \exists x\,(\exists vw\,(Zvab \wedge Zwac \wedge Sxvw) \wedge u \sqsubseteq x)))\end{aligned}$$

is derivable in (Ax 5), (Ax 14), (Ax 15), (Ax FUS). Therefore, with the definition of 'Δ' in the second step:

(Ax 5, 14, 15, FUS) $\vdash Pa \wedge Pb \wedge Pc \wedge \neg Coll(abc) \rightarrow Pa \wedge Pb \wedge Pc \wedge \neg Coll(abc) \wedge \exists y \forall u\, (Pu \rightarrow (u \sqsubseteq y \leftrightarrow \exists xvw\, (Zvab \wedge Zwac \wedge Sxvw \wedge u \sqsubseteq x)))$
$\rightarrow \exists y\, \Delta yabc$

With circles, there is a similar situation. Like 'Δ,' 'K' can be defined in such a way that the relevant axiom (Ax 19) can be mereologically derived. Moreover, the revised definition can also be used to prove all the results we have proved. Thus, let us define 'K' using only expressions which belong to L[E$^-$].

Definition: $Kyab :\leftrightarrow Pa \wedge Pb \wedge a \neq b \wedge$
$\forall u\, (Pu \rightarrow (u \sqsubseteq y \leftrightarrow \exists st\, (Ssab \wedge Stau \wedge s \equiv t)))$

Lemma 9: (i) (Ax 5, 8) $\vdash \forall xyz\, (Kxyz \rightarrow z \sqsubseteq x)$
(ii) (Ax 2) $\vdash \forall xyz\, (Kxyz \rightarrow \forall w\, (Pw \wedge w \sqsubseteq x \rightarrow y \neq w))$
(iii) (Ax 4) $\vdash \forall xyz\, (Kxyz \rightarrow \forall wss'\, (Pw \wedge w \sqsubseteq x \wedge Ssyz \wedge Ss'yw \rightarrow s \equiv s'))$
Proof: (i) By (Ax 5) and (Ax 8), we have:

(Ax 5, 8) $\vdash Py \wedge Pz \wedge y \neq z \rightarrow \exists s\, (Ssyz \wedge Ssyz \wedge s \equiv s)$
$\rightarrow \exists st\, (Ssyz \wedge Styz \wedge s \equiv t)$

By the definition of 'K,' it follows:

(Ax 5, 8) $\vdash Kxyz$
$\rightarrow Pz \wedge \forall u (Pu \rightarrow (u \sqsubseteq x \leftrightarrow \exists st (Ssyz \wedge Styu \wedge s \equiv t))) \wedge \exists st (Ssyz \wedge Styz \wedge s \equiv t)$
$\rightarrow (z \sqsubseteq x \leftrightarrow \exists st\, (Ssyz \wedge Styz \wedge s \equiv t)) \wedge \exists st\, (Ssyz \wedge Styz \wedge s \equiv t)$
$\rightarrow z \sqsubseteq x$

(ii) By the definition of 'K' and with (Ax 2) in the second step, we have:

(Ax 2) $\vdash Kxyz \wedge Pw \wedge w \sqsubseteq x \rightarrow \exists st\, (Ssyz \wedge Styw \wedge s \equiv t)$
$\rightarrow y \neq w$

This implies the claim sought.

(iii) By the definition of 'K' and with (Ax 4) in the second step, we have:

(Ax 4) $\vdash Kxyz \wedge Pw \wedge w \sqsubseteq x \wedge Ssyz \wedge Ss'yw$
$\rightarrow \exists uv\, (Suyz \wedge Svyw \wedge u \equiv v \wedge Ssyz \wedge Ss'yw)$
$\rightarrow \exists uv\, (u = s \wedge v = s' \wedge u \equiv v)$
$\rightarrow s \equiv s'$

This implies the claim to be proven.

Lemma 10: (Ax 8, FUS) $\vdash$ (Ax 19)
Proof: Let ψ be the L[E$^-$] formula

$\exists st\, (Ssab \wedge Stax \wedge s \equiv t \wedge Px)$.

Then:

$$\vdash Pa \wedge Pb \wedge Ssab \wedge Ssab \wedge s \equiv s \wedge Pb \rightarrow \exists st\,(Ssab \wedge Stab \wedge s \equiv t \wedge Pb) \\ \rightarrow \exists x \exists st\,(Ssab \wedge Stax \wedge s \equiv t \wedge Px) \\ \rightarrow \exists x\, \psi$$

Hence it follows by (Ax 8) and (Ax FUS):

$$\text{(Ax 8, FUS)} \vdash Pa \wedge Pb \wedge Ssab \\ \rightarrow \exists y \forall u\,(Pu \rightarrow (u \sqsubseteq y \leftrightarrow \exists x\,(\exists st\,(Ssab \wedge Stax \wedge s \equiv t \wedge Px) \wedge u \sqsubseteq x)))$$

Therefore, employing the definition of 'Px' in the second step:

$$\text{(Ax 8, FUS)} \vdash Pa \wedge Pb \wedge Ssab \\ \rightarrow \exists y \forall u\,(Pu \rightarrow (u \sqsubseteq y \leftrightarrow \exists st\,(Ssab \wedge s \equiv t \wedge \exists x\,(Stax \wedge Px \wedge u \sqsubseteq x)))) \\ \rightarrow \exists y \forall u\,(Pu \rightarrow (u \sqsubseteq y \leftrightarrow \exists st\,(Ssab \wedge s \equiv t \wedge \exists x\,(Stax \wedge Px \wedge u = x)))) \\ \rightarrow \exists y \forall u\,(Pu \rightarrow (u \sqsubseteq y \leftrightarrow \exists st\,(Ssab \wedge s \equiv t \wedge Stau)))$$

It follows that:

$$\text{(Ax 8, FUS)} \vdash Pa \wedge Pb \wedge Ssab \\ \rightarrow Pa \wedge Pb \wedge Ssab \wedge \exists y \forall u\,(Pu \rightarrow (u \sqsubseteq y \leftrightarrow \exists st\,(Ssab \wedge Stau \wedge s \equiv t))) \\ \rightarrow \exists y\, Kyab.$$

5 Mereogeometry

What sort of a theory is EG? On the intended interpretation of L[E], '$\sqsubseteq$' is clearly mereological vocabulary. We would probably classify the other primitive predicates as geometric. These predicates are certainly often found in texts on geometry. Thus EG is a theory that contains both geometric and mereological vocabulary and moreover connects the geometric and mereological terms. On analogy with the now popular term 'mereotopology' (for the combination of mereological and topological principles), we suggest calling such a theory a *mereogeometry*. EG would be a mereogeometrical theory.

We admit that the term 'mereogeometry' is only as clear as the terms 'mereological theory' and 'geometrical theory'. And it is not clear how to explicate those predicates. It should not depend on the notation of the formal language. The decisive matter is the intended interpretation of the vocabulary and the axioms that govern its use in the theory. But these two factors (interpretation and axioms) do not yield entirely clear criteria for distinguishing mereological and geometric theories.

Moreover, even if it is *prima facie* clear that a predicate belongs to the geometric and not to the mereological vocabulary, this might not be true of a reconstruction or explication of that predicate. One example of this is Euclid's own definition of 'point' as 'that which has no part' (I, def. 1). This *definiens* contains only mereological vocabulary.

5.1 Can the mereological vocabulary of EG be eliminated?

Our main concern, however, is whether the mereological content of our theory EG might be eliminated by defining the mereological term in purely geometric terms. This would show that our theory is, in a sense, not 'essentially' mereological.

One might attempt to define '$\sqsubseteq$' in terms of the geometric predicates of L[E], but we do not see any way to do this. Alternatively, one might introduce a further primitive two-place predicate, I, to be read '…intersects …'. (We will use the term such that a point that lies on a line intersects that line, and the line intersects the point.) Then '$\sqsubseteq$' can be removed from the primitive vocabulary of L[E] and defined as follows:

Definition: $x \sqsubseteq y :\leftrightarrow \forall z\,(xIz \rightarrow yIz)$

(Ax 11) and (Ax 12) are then consequences of this definition and become theorems. (Ax 15), however, is still required. (It can be reformulated in terms of 'I'.)

Is the predicate 'I' a geometric one or itself mereological? In light of the considerations given just above, this question resolves itself into two other questions: (1) Is 'intersects' a geometric or a mereological predicate? (2) Are the axioms for 'I' best interpreted geometrically or mereologically? Neither question seems answerable. The first is not answerable, because our linguistic intuitions do not even yield the intuition that the one or other answer is wrong. The second is not answerable because, except for (Ax 15), we need no axioms for 'I' and (Ax 15) seems to be neither distinctively geometric nor distinctively mereological. In particular, one might interpret 'I' also as 'overlap,' which poses no problem for (Ax 15), and 'overlap' is generally taken to be paradigmatic mereological vocabulary.

In short, there seems no reason to think that the mereological vocabulary of theory EG could be defined in purely geometric terms and thereby eliminated.

5.2 Extended geometric objects

Extended objects, such as triangles and circles (but also straight lines), present another problem for the distinction between mereological and geometric vocabulary. Such extended objects might be construed as being composed of points. But we would then expect that the predicates that refer to them do not belong to purely geometric vocabulary. This requires some explanation.

A straight line, s, is determined by two endpoints, a and b. But s is distinct from a and from b. Analogously, a triangle t is determined by three points or by three straight lines, but it is distinct from each of the three points and from each of the straight lines. Likewise, a circle is determined by two points, the center and any arbitrary point on its circumference, but it is distinct from each of these points.

Thus one might take the straight line from a to b to be the set of all points that are collinear with a and b and between them. Or to be simply the ordered pair (a, b). Or to be the two-member set $\{a, b\}$. And the triangle with vertices a, b, and c might be the ordered triple (a,b,c) or some other set-theoretic construct involving a, b, and c. Finally, it would be natural to take a circle k around a through b to be the set of points which have the same distance from a as b. But these are set-theoretic definitions and constructions, which contemporary mathematics is accustomed to use, but Euclid did not think in this way. As interpreters of Euclid and as logicians who are attempting (for the moment) to avoid set theory, neither should we.

Intuitively, if a triangle or a circle or straight line is any kind of construction out of points, then it is not a set, but rather a mereological fusion. That is, it is a whole that has the relevant points as parts.

Moreover, we need to address the question whether circles, triangles, etc. include only their boundary points or also their interior points? For Euclid, the interior points of circles, triangles, etc. belong to those figures. Indeed, he defines figure (or shape; σχῆμα) as "what is contained [περιεχόμενον] by some boundary or boundaries [ὅρων]" (I, def. 4). The figure is not the boundary, but that which is contained by the boundary. This conception is reiterated then in the definitions of the various figures, each of which is defined as that which is contained by a certain sort of boundary (see Book I, defs. 15, 18, and 19).

While we spoke from time to time as if triangles and circles were to be identified with their boundaries, this is a mere matter of terminology. 'Δ' can be interpreted either as the predicate '...is a triangle [Euclidean figure] determined by ..., ..., and ..' or as the predicate '...is the perimeter of a triangle determined by ..., ..., and ..' With either interpretation, our existence axiom (Ax 18) will come out true. Something similar is true for 'K,' but it is just slightly more complicated. As far as our existence axiom (Ax 19) is concerned, 'K' might just as well be the predicate '...is a circle [Euclidean figure] around ...through ..' However, our definition of 'K' rules this out and is compatible only with interpreting 'K' as a circumference predicate. This, however, is easily avoided. The definition of 'K' could be reformulated as a definition of another predicate, say '$\check{K}$,' so that 'K' is a predicate for circles (Euclidean figures with interiors) whereas '$\check{K}$' is a predicate for a circle's circumference.

One might also consider giving no definition at all for 'K.' The use of the predicate would then have to be governed entirely by axioms. An obvious way to do this would to add Lemma 9 (i), (ii) and (iii) (along with (Ax 19)). As already noted, all of the propositions proved here could be proven on this basis.

In this case, neither 'Δ' nor 'K' would be defined using mereological vocabulary. Triangles and circles would thus not have been introduced as mereological fusions. Other axioms could entail that they are mereological fusions, but not the axioms of EG. Even if circles and triangles are in fact fusions (of points, say), our theory would simply ignore this fact.

However, none of this would significantly affect the role of mereology in EG. For the existence axioms for triangles and circles (and also Lemma 9 (i), (ii), and (iii), which would become axioms on this scenario) use mereological vocabulary. Mereology and mereogeometry are thus used whenever these axioms are used, even if we refrain from defining 'Δ' and 'K.'

5.3 On the usefulness of mereogeometry

How is mereogeometry useful and interesting? Here we should distinguish between a number of related questions.

First, does Euclid's *Elements* present a mereogeomtry? Our consideration both of Book II and of Book I, Propositions 1 through 10 (without 7) gives strong evidence that the answer is yes. This will not be a great surprise to interpreters of Euclid. Our contribution has two aspects. We have showed how, in connection with Book II, it is mereological concepts (part, whole, composition, dissection) that play the role we might have expected quantitative operations (addition and subtraction of real numbers) to play. This concerns not merely the question whether mereology plays a role, but what role it plays. Also, we have showed carefully and specifically how, without relying on diagrams, mereology enters the initial propositions of Book I.

Second, is it possible to formulate a geometric theory in a first-order logical language, which contains neither set-theoretic nor mereological vocabulary, and only predicates that apply to n-tuples of points? Here, again, the answer is yes. This has been known since Tarskis investigations of the foundations of geometry in the middle of the 20th century.[11] In particular, Tarski specified a language without predicates that express '...is a straight line,' '...is a triangle,' or '...is a circle.' It is, however, also not clear how these predicates might be defined within Tarski's framework. The familiar way of speaking

11 See Schwabhäuser, Szmielew, and Tarski 1983.

about geometric figures must be replaced, within Tarski's framework, by talk about n-tuples of points (e.g., a triangle is treated as a triple of non-collinear points).

Third, supposing that one wants to speak of extended geometric objects (by contrast with n-tuples of points), how should one do this? It is something that we wish to do, as we explained in the Introduction, partly as an interpretation of Euclid, partly for its intrinsic philosophical interest. There seem two main approaches. Either the relevant predicates are introduced via definitions or their use is governed by axioms. In the first case, it would be natural to use mereological vocabulary in the *definientia*. This seems to us the preferable approach. This is not because we consider definitions to be superior to axioms. (It is not at all clear that the approach reduces the total number of axioms, for axioms are also needed to govern the mereological primitives.) Rather, we have used a generally accepted mereological fusion schema. In section 4.3, we saw how we can use this axiom (together with Atomism, which in any case fits well with the geometry of EG) to derive (Ax 18) and (Ax 19). Thus the advantage of this approach is the way it enables us to avoid postulating highly specific geometric axioms. This approach seems to us well worth developing, because we think (with Euclid?) that we ordinarily speak of extended objects. And we think that such a logic can contribute to clarifying both Euclid's theory and the systematic philosophical questions about how to understand extended objects.

5.4 The way forward

This paper is contains significant results and they invite further research. First, and perhaps most obviously, we should complete our formalization of Euclid Books I and II. (It would also be desirable to consider whether the axioms that are sufficient for Books I and II are also sufficient for III and IV. This seems likely, but it is not entirely obvious.) This would be a sort of commentary on the relevant parts of the *Elements.* As noted, it seems to us important to have such a logical treatment of the *Elements* that does not use the diagrams for drawing inferences. Second, our approach to geometry should be compared in details with Tarski's. Most significantly, can a logical language of *extended* objects (by contrast with n-tuples of points) be developed using mereology that has essentially greater expressive power? Third, we would like to refine our approach by reconsidering the concept of a boundary and its relation to the concept of a part. We have treated points as parts of lines and lines as parts of figures. It is not so clear that Euclid does this. In Section 2, we found Euclid treating *figures* as parts of figures. It is likely that Euclid, in fact, thinks that only objects of same kind can stand in the part-whole relation.[12] This would require us to extend L[E] by at least one additional predicate. It would also bring our work into connection with recent mereotopology,[13] from which we would hope to benefit, and to which we would hope to make a contribution. In particular, Euclid's theory seems to be one with points (which have no parts) but without atoms (because lines are not composed of points, nor figures of lines). Aristotle also espoused such a theory. But current philosophers and logicians, although they are interested in geometries without atoms, have not, to our knowledge, considered such theories.

12 One might think that 'of the same kind' means simply 'with the same number of dimensions,' but this is probably too quick. An angle cannot be part of a figure, but an angle seems to be a two-dimensional object. Having the same number of dimensions is probably a necessary, but not sufficient, condition on standing in the part-whole relation.

13 E.g., Glibowski 1969; Simons 1987; Smith 1997; Casati and Varzi 1999.

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